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Divergence of a quantum thermal state on Kerr space-time

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Abstract

We present a simple proof, using the conservation equations, that any quantum stress tensor on Kerr space-time which is isotropic in a frame which rotates rigidly with the angular velocity of the event horizon must be divergent at the velocity of light surface. We comment on our result in the light of the absence of a 'true Hartle-Hawking' vacuum for Kerr.

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One of the fundamental results of quantum field theory in Kerr space-time is a theorem of Kay and Wald [1], that there does not exist a Hadamard state which respects the symmetries of the space-time and is regular everywhere outside and on the event horizon. This means that there is no 'true Hartle-Hawking' (HH) vacuum on Kerr space-time. The HH vacuum on Schwarzschild black holes has been extensively studied, since it is the state which lends itself most readily to numerical computations, due to its regularity and high degree of symmetry. For the same reason the construction of a state on Kerr space-time with most (but not all) of the properties of the HH state remains an important open question.

The construction of quantum states on Kerr space-time is a delicate matter due to the presence of the super-radiant modes (see [2] for details of this procedure). There is a consensus in the literature that the (past) Boulware vacuum $|B^-\rangle$ is defined by taking a basis of modes which are positive frequency with respect to the Killing vector $\partial/\partial t$ at $\mathfrak{I}^-$ and with respect to the Killing vector $\partial/\partial t + \Omega_H \partial/\partial\phi$ (where Ω_H is the angular velocity of the horizon) at $\mathfrak{H}^-$. This state corresponds to an absence of particles coming up from $\mathfrak{H}^-$ or in from $\mathfrak{I}^-$, and contains, at $\mathfrak{I}^+$, an outward flux of particles due to the Unruh-Starobinskii effect (spontaneous emission in superradiant modes). There are two attempts in the literature to define a state analogous to the HH state [3,4], one due to Frolov and Thorne [3] and the other due to Candelas, Chrzanowski and Howard [4] which we shall denote by $|FT\rangle$ and $|CCH\rangle$, respectively. We refer the reader to [2] for details of the construction of these states, which are not important here. Both these candidate states are thermal in nature (although the energy with respect to which the modes are thermalized differ), but they have different symmetry and regularity properties. The state $|FT\rangle$ is invariant under simultaneous t, ϕ reversal, while $|CCH\rangle$ is not.

Our interest in this letter is in the difference in expectation values of the quantum stress tensor in the Boulware and candidate HH vacua. By considering such a difference any renormalization terms cancel and we may deal effectively deal with the bare operators. We shall denote this tensor as $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ regardless of which candidate HH vacuum we are considering, since our result in this paper is quite general and does not depend on the details of the construction of the thermal states. In [3], $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ represents a precisely thermal atmosphere of quanta at the Hawking temperature which rotates rigidly with the same angular velocity as the event horizon, Ω_H. Therefore, $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ should be isotropic in a frame which also rotates rigidly with angular velocity Ω_H. This is in agreement with the calculation of [4], where the difference in expectation values of the stress tensor in the Boulware $|B^-\rangle$ and $|CCH\rangle$ states was calculated at the event horizon for an electromagnetic field. They found that this tensor was isotropic in the Carter tetrad [5]. Due to the rotation of the black hole, this is the same, at the event horizon, as the tensor being isotropic in a tetrad which is rigidly rotating with the same angular velocity as the black hole. The calculation in [4] is valid only close to the event horizon, and does not provide any information about the isotropy (or otherwise) of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ away from the horizon.

Frolov and Thorne [3] then used an argument based on the details of the quantum field modes to show that this stress tensor will fail to be regular at the velocity of light surface $\mathcal{S}_L$. This is the surface, where, in order to co-rotate with the event horizon, an observer must travel at the speed of light. We shall now show that it is an elementary consequence of the conservation equations that $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$, with the assumed isotropy property, will fail to be regular at $\mathcal{S}_L$. In addition, we will also find that $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ has the same, divergent, form

at the event horizon as found by [4]. Our metric has signature $(-+++)$ and we use units in which $G=c=\hbar=k_b=1$ throughout. Greek letters will denote co-ordinate components, while bracketed Roman letters denote tetrad components.

Firstly, we begin by writing the Kerr metric in the unconventional form [3]

$$ds^2 = -\alpha^2 dt^2 + \rho^2 \Delta^{-1} dr^2 + \rho^2\, d\theta^2 + \tilde{\Omega}^2 (d\phi - \Omega\, dt)^2, \tag{1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, with M the mass of the black hole and a the angular momentum per unit mass, as viewed from infinity. The other functions appearing in the metric (1) are given by:

$$\begin{aligned}
\alpha^2 &= \frac{\Delta \rho^2}{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}, \\
\Omega &= \frac{2Mra}{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}, \\
\tilde{\Omega}^2 &= \frac{1}{\rho^2}\left[\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta\right]\sin^2\theta.
\end{aligned} \tag{2}$$

Here α is the lapse function, vanishing on the event horizon $r = r_H$ (when $\Delta = 0$), and Ω is the angular velocity of a locally non-rotating observer (LNRO), which is equal to the angular velocity Ω_H of the event horizon when $r = r_H$. An observer who is rotating with angular velocity Ω_H has the Lorentz factor γ relative to a LNRO at the same values of r and θ, where

$$\gamma = \left(1-v^2\right)^{-\frac{1}{2}}, \qquad v = \alpha^{-1}\left(\Omega_H - \Omega\right)\tilde{\Omega}. \tag{3}$$

At the event horizon, an LNRO has angular velocity Ω_H and in this case $\gamma = 1$, whilst at $\mathcal{S}_L$, $v \to 1$ and $\gamma \to \infty$, as expected.

An orthonormal frame which rotates rigidly with angular velocity Ω_H has basis 1-forms given by:

$$\begin{aligned}
e_{(t)i}dx^i &= \alpha^{-1}\gamma\left[\alpha^2 dt - \tilde{\Omega}^2(\Omega_H - \Omega)(d\phi - \Omega\, dt)\right], \\
e_{(r)i}dx^i &= (\rho^2\Delta^{-1})^{\frac{1}{2}} dr, \\
e_{(\theta)i}dx^i &= \rho\, d\theta, \\
e_{(\phi)i}dx^i &= \tilde{\Omega}\gamma(d\phi - \Omega_H\, dt).
\end{aligned} \tag{4}$$

We shall consider a stress tensor $\langle \hat{T}_{\mu\nu}\rangle_{HH-B}$ which is isotropic in this frame, so that with respect to the basis of 1-forms (4) the tetrad components are:

$$\langle \hat{T}^{(a)}_{(b)}\rangle_{HH-B} = f(r,\theta)\,\mathrm{diag}\left\{-3,1,1,1\right\}, \tag{5}$$

where we have used the Killing vector symmetries of the geometry to assume that $\langle \hat{T}_{\mu\nu}\rangle_{HH-B}$ does not depend on either t or ϕ [2]. In order to find the unknown function $f(r,\theta)$, we solve the conservation equations for $\langle \hat{T}_{\mu\nu}\rangle_{HH-B}$. The simplest way to do this, which avoids the

calculation of Ricci rotation coefficients for the tetrad (4), is to convert the tetrad components back to Boyer-Lindquist co-ordinate components and then solve the conservation equations in the form [2]:

$$\partial_\nu \left(\langle \hat{T}^\nu_\mu \rangle_{HH-B} \sqrt{-g} \right) = \frac{1}{2} \sqrt{-g} \left(\partial_\mu g_{\lambda\sigma} \right) \langle \hat{T}^{\lambda\sigma} \rangle_{HH-B}. \tag{6}$$

The $\mu = t$ and $\mu = \phi$ equations are trivial, and the $\mu = r$ and $\mu = \theta$ equations give, respectively,

$$\begin{aligned} \partial_r \left(f(r,\theta) \rho^2 \sin\theta \right) &= \frac{1}{2} f(r,\theta) \rho^2 \sin\theta \, \partial_r \left(\log |\alpha^{-8} \gamma^8 \rho^4 \sin^2\theta| \right) \\ \partial_\theta \left(f(r,\theta) \rho^2 \sin\theta \right) &= \frac{1}{2} f(r,\theta) \rho^2 \sin\theta \, \partial_\theta \left(\log |\alpha^{-8} \gamma^8 \rho^4 \sin^2\theta| \right). \end{aligned} \tag{7}$$

These two equations are compatible and determine $f(r,\theta)$ up to an arbitrary constant. The result is

$$f(r,\theta) = k \alpha^{-4} \gamma^4 \tag{8}$$

where k is an arbitrary constant. This implies that the tetrad components of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ diverge as Δ^{-2} as the event horizon is approached, which is the behaviour found in [4].

In order to consider the regularity (or otherwise) of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ at the event horizon or $\mathcal{S}_L$, we must first convert the tetrad components to Boyer-Lindquist components, since the tetrad (4) is not regular either at the event horizon or $\mathcal{S}_L$. The non-zero components of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ are:

$$\begin{aligned} \langle \hat{T}_{tt} \rangle_{HH-B} &= \left[3\alpha^2 + \tilde{\Omega}^2 \left(4\gamma^2 \Omega_H^2 - 3\Omega^2 \right) \right] f(r,\theta), \\ \langle \hat{T}_{t\phi} \rangle_{HH-B} &= -\tilde{\Omega}^2 \left(4\gamma^2 \Omega_H - 3\Omega \right) f(r,\theta), \\ \langle \hat{T}_{\phi\phi} \rangle_{HH-B} &= \tilde{\Omega}^2 \left(4\gamma^2 - 3 \right) f(r,\theta), \\ \langle \hat{T}_{rr} \rangle_{HH-B} &= \rho^2 \Delta^{-1} f(r,\theta), \\ \langle \hat{T}_{\theta\theta} \rangle_{HH-B} &= \rho^2 f(r,\theta). \end{aligned} \tag{9}$$

Using these components, the regularity of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ on the event horizon can be considered by first transforming to Kruskal co-ordinates, as in [2]. Considering the Kruskal components shows that $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ is not regular on either the past or the future event horizon, due to the Δ^{-2} divergence of f as the horizon is approached. This is precisely the behaviour expected of the Boulware vacuum close to the event horizon, and does not preclude the possibility that a candidate HH state may be regular on some section of the event horizon. In [2] it was argued that, of the two candidates for the analogue of the HH vacuum in Kerr, $|FT\rangle$ is regular only at the pole of the event horizon, and $|CCH\rangle$ is regular on the future (but not on the past) event horizon. Both these cases are compatible with our results here, provided that the divergences (where they exist) are of lower order than the Δ^{-2} expected for the Boulware vacuum.

The Boyer-Lindquist co-ordinate system is regular at $\mathcal{S}_L$, and so (9) reveals that the components of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ diverge at least as fast as γ^4 as the velocity of light surface is

approached and $\gamma \to \infty$. Thus, if $\langle B^- | \hat{T}_{\mu\nu} | B^- \rangle_{ren}$ is regular at the velocity of light surface, then the expectation value of the stress tensor in the HH vacuum diverges there. This is certainly the case for slowly rotating black holes when $\mathcal{S}_L$ is far from the horizon and $\langle B^- | \hat{T}_{\mu\nu} | B^- \rangle_{ren}$ is given by the Unruh-Starobinskii effect. In fact, the Boulware vacuum is expected to be regular everywhere away from the event horizon (where it diverges). We conclude that *if* we have a state $|H\rangle$ such that $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ is isotropic in the tetrad (4), then the state $|H\rangle$ will not be regular on $\mathcal{S}_L$.

Thus we have shown that a stress tensor which is isotropic with respect to a frame which rotates rigidly with the angular velocity of the event horizon must diverge as Δ^{-2} as the event horizon is approached, and must also fail to be regular at $\mathcal{S}_L$. The question is therefore, is this isotropy condition likely to be satisfied by a physical stress tensor? The answer is far from obvious as the peak of the thermal spectrum at the Hawking temperature is at a wavelength comparable to the radius of curvature of the space-time near the horizon. In the absence of a full numerical calculation in Kerr, we note that the corresponding property of $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ for Schwarzschild black holes was conjectured in [6], and subsequently shown to hold to a good approximation in [7]. The authors of [3] used the thermal properties of their state $|FT\rangle$ to show that the corresponding stress tensor $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ is isotropic in the rigidly rotating tetrad, although in [2] we have argued that, despite its attractive symmetry properties (in particular, simultaneous t, ϕ reversal invariance) this state is fundamentally flawed. The alternative state $|CCH\rangle$, which is not invariant under simultaneous t, ϕ reversal, is more workable, and satisfies the required isotropy property, at least near the event horizon [4]. Note that, in contrast to the situation in Schwarzschild, the Boulware vacuum in Kerr is not invariant under simultaneous t, ϕ reversal. Therefore, we would expect any analogue HH state such that $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ has our conjectured isotropy must also not be t, ϕ reversal invariant.

It should be stressed that it is crucial to our analysis that the stress tensor $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ is isotropic in the rigidly rotating frame (4) rather than any other tetrad. For example, a stress tensor which is isotropic with respect to the Carter tetrad [5] everywhere outside the event horizon will also display the divergence near the event horizon we have exhibited here, but will be regular elsewhere. This would be anticipated from the fact that the Carter tetrad rotates with an angular velocity which, although not the same as the angular velocity of the LNROs, decreases as we move away from the event horizon. Therefore, observers who are rotating with the same angular velocity as the Carter tetrad will always have a finite Lorentz factor relative to the LNROs (compare (3)). The point of [3] is that only observers who have angular velocity Ω_H (whatever their position outside the event horizon) will see an isotropic thermal distribution of particles, which is the reason for our conjectured isotropy condition here. The argument in [3] uses the thermal properties of their state $|FT\rangle$, and does not apply to the other candidate HH state, $|CCH\rangle$, due to the different thermalization of the modes [2]. We hope to return to this question subsequently, by examining the isotropy (or otherwise) of these candidate HH vacua using numerical calculations.

In the light of the Kay-Wald theorem [1], which states that there is no true HH state on Kerr spacetime, our result shows that, if it is possible to define a state on Kerr which has most of the properties of the HH state, then either the stress tensor $\langle \hat{T}_{\mu\nu} \rangle_{HH-B}$ corresponding to this state will fail to be isotropic in the rigidly rotating tetrad, or the state will cease to be regular on $\mathcal{S}_L$. Furthermore, we have shown that the divergence at $\mathcal{S}_L$ is an elementary

consequence of the conservation equations.

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